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The radiated fields of multipole point sources near a solid spherical surface

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The acoustic field radiated by a multipole point source positioned near to the surface of a solid sphere is calculated at both low and high frequencies. It is shown that the scattered field at low frequencies is always dipole, but at high frequencies is of the same type as the incident field. The application of the results to the acoustic field radiated by turbulence near a sphere is briefly discussed.

Introduction

This paper considers the scattering of an acoustic field by a solid body in a fluid of infinite extent. The incident field is that due to simple or multipole point sources. High- and low-frequency limits are calculated.

If the scattering body is of a shape suitable for a co-ordinate system which admits separable solutions of the wave equation, the scattered field can, in general, be written as an infinite summation of eigenfunctions. It is usually possible to express the incident field as a sum over the same eigenfunctions. Thus, specifying a suitable boundary condition at the scattering surface, an exact solution can be obtained. However, this infinite summation is often not a particularly useful description of the acoustic field. Simpler descriptions showing the physical nature of the field can be obtained in some cases of which examples of scattering by a sphere are discussed here.

The low-frequency limit is first calculated using Kirchhoff's integral over the surface pressure. It is easily shown that for any smooth rigid body the scattered field is always dipole. The relative strength of the scattered field is calculated explicitly for scattering by a sphere of the incident field due to a simple source. Results for multipole sources can be obtained immediately from the simple source solution by differentiation with respect to the source position variable. The dipole form of the scattered field exists irrespective of the properties of the incident field.

At high frequencies this dipole dominance does not occur. The limiting case is specular reflexion. The dipole then disappears completely and the plane merely reflects the incident field. For bodies with large radius of curvature, that is, for sound fields with wavelengths small compared to the dimensions of the body, we would again expect that the function of the scattering body was merely to reflect the incident field, although with some distortion. A high-frequency approximation obtained by tracing acoustic rays is given in this paper for an incident field due to a simple source. Again results for dipole and quadrupole fields are obtained. It is known that at this limit the scattered field is essentially of the same type as the incident field.

The results are used to compare features of the noise radiated by a turbulent boundary layer on a rigid sphere at high and low frequencies. It is shown that the parameter MS, where M is the Mach number of the mean flow and S is a Strouhal number, determines which of the limiting cases is relevant.

The low-frequency limit is also used here to determine the field radiated by a dipole physically attached to a submerged spherical body. The sphere oscillates under the influence of both fluid and mechanical coupling to the dipole, and itself generates a dipole field. The total field is obtained by simple addition and it is shown that the resulting radiation is least when the dipole is rigidly attached.

The series expansion of the total field

The field due to a simple source near to a rigid fixed sphere is first calculated. Separable solutions of the wave equation in spherical co-ordinates (r, θ, ϕ) for waves travelling outwards at infinity can be written as the infinite summation (with an implied $e^{i\omega t}$ dependence)

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{mn} P_n^m(\cos\theta) h_n(kr) \cos m(\phi - \phi_0), \tag{1}$$

where $k = \omega/c$ is the wave-number of the field; $h_n(kr)$ denotes a spherical Hankel function of the first kind (the usual superscript is omitted) and $P_n^m(\cos\theta)$ denotes an associated Legendre function. The terms α_{mn} are to be determined by the boundary condition.

The acoustic pressure field $p_i(\mathbf{r})$ at the point $\mathbf{r}$ due to a unit simple source at $\mathbf{r}_0$ can be expressed in spherical co-ordinates in the same functions as expression (1) (Morse & Feshbach 1953).

$$p_i(\mathbf{r}) = \frac{e^{ikR_0}}{R_0} = ik \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{\infty} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos m(\phi - \phi_0) P_n^m(\cos\theta)\, P_n^m(\cos\theta_0)$$

$$\times \begin{cases} j_n(kr_0) h_n(kr) & (r > r_0), \\ h_n(kr_0) j_n(kr) & (r < r_0), \end{cases}$$

where $R_0 = |\mathbf{r} - \mathbf{r}_0|$, j_n denotes a spherical Bessel function, $\epsilon_m = 1$ if $m = 0$ and $\epsilon_m = 2$ if $m > 0$. The appropriate factor is to be taken as either $r > r_0$ or $r < r_0$.

The boundary condition to be satisfied at the surface is that the normal velocity there vanish. This is equivalent to the condition

$$\partial p(\mathbf{r})/\partial r|_{r=a} = 0,$$

where a is the radius of the sphere. p denotes the total acoustic pressure obtained by adding the scattered field p_s to the incident source field p_i. If p_s is expanded as in (1), the boundary condition can be used to obtain an equation for the acoustic pressure p.

$$p(\mathbf{r}) = ik \sum_n (2n+1) \sum_m \epsilon_m \frac{(n-m)!}{(n+m)!} \cos m(\phi-\phi_0) P_n^m(\cos\theta) P_n^m(\cos\theta_0)$$
$$\times \begin{cases} \dfrac{h_n(kr)}{h_n'(ka)} \{j_n(kr_0) h_n'(ka) - h_n(kr_0) j_n'(ka)\} & (r > r_0), \\ \dfrac{h_n(kr_0)}{h_n'(ka)} \{j_n(kr) h_n'(ka) - h_n(kr) j_n'(ka)\} & (r < r_0). \end{cases} \tag{2}$$

The dash denotes differentiation with respect to the argument. Modifications of this result for source fields due to point multipoles can easily be found. In Cartesian co-ordinates these fields are equivalent to differentiating the simple source field with respect to the source position variable. General expressions for point multipoles in co-ordinates other than Cartesian are cumbersome because of the curvature of the co-ordinate system. Here we consider dipoles and longitudinal quadrupoles corresponding to $\partial/\partial y_3$ and $\partial^2/\partial y_3^2$, respectively, where y_3 corresponds to the axis $\theta = 0$, at two source positions, $(r_0, 0, 0)$ and $(r_0, \frac{1}{2}\pi, 0)$ respectively. Direct comparison is thus possible between longitudinal multipoles inclined in the radial and circumferential directions. We note that as the boundary condition at the sphere is homogeneous the differentiation can be applied directly to the total incident and scattered field.

In spherical co-ordinates, $\partial/\partial y_3$ can be represented as

$$\frac{\partial}{\partial y_3} = \cos\theta_0 \frac{\partial}{\partial r_0} - \sin\theta_0 \frac{1}{r} \frac{\partial}{\partial \theta_0}.$$

Thus, at the source point $(r_0, 0, 0)$, the required dipole and quadrupole results are obtained from $\partial/\partial r_0$ and $\partial^2/\partial r_0^2$, respectively, of the simple source field.

At the source point $(r_0, \frac{1}{2}\pi, 0)$, the required dipole and quadrupole results are obtained from $-(1/r_0)\,\partial/\partial\theta_0$ and $-(1/r_0^2)\,\partial^2/\partial\theta_0^2 - (1/r_0)\,\partial/\partial r_0$, respectively, of the simple source field. The minus sign is included here to keep the same sense of the multipole.

The differentiation to obtain multipole results is to be applied in (2). However, the summation in this expression cannot be performed directly. High- and low-frequency approximations to the distant pressure field will be calculated. As these approximations are made independently of the source position, the differentiation for multipole fields can again be applied directly to the simple source result so obtained.

Low-frequency limit

The low-frequency approximation is obtained from the Kirchhoff result for the field in terms of the integral of its value at the surface. For a field with a vanishing normal derivative at the surface, the total field can be written in the form

$$p(\mathbf{r}) = p_i(\mathbf{r}) + \frac{1}{4\pi} \int_s p(\mathbf{r}_1) \frac{\partial}{\partial a} \left(\frac{e^{ikR_1}}{R_1} \right) d\mathbf{S}. \tag{3}$$

The point $\mathbf{r}_1$ lies on the surface of the sphere and

$$R_1^2 = |\mathbf{r}-\mathbf{r}_1|^2 = r^2+a^2-2ar\cos\psi,$$

where $\cos\psi = \cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos(\phi-\phi_1)$. The exponential term in (3) has a simple form at small values of ka if a far-field approximation is taken (i.e. if $ka \ll 1$ and $a \ll r$). Equation (3) can be written in this case

$$p(\mathbf{r}) = p_i(\mathbf{r})-\frac{ik}{4\pi}\frac{e^{ikr}}{r}\int_s p(\mathbf{r}_1)\cos\psi\, d\mathbf{S}. \tag{4}$$

The pressure at the surface can be written in the form

$$p(\mathbf{r}_1) = \frac{-k}{(ka)^2}\sum_n (2n+1)\sum_m \epsilon_m \frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\theta_0)\cos m(\phi-\phi_0)\frac{h_n(kr_0)}{h_n'(ka)},$$

where we have used the relation

$$j_n(ka)\,h_n'(ka)-j_n'(ka)\,h_n(ka) = i/(ka)^2.$$

The integral in (4) is now performed by noting that

$$\cos\psi = \cos\theta P_1^0(\cos\theta_1)+\sin\theta P_1^1(\cos\theta_1)\cos(\phi-\phi_1),$$

and using the orthogonality properties of the associated Legendre functions. Only one term of the summation gives a non-zero result. The scattered field is

$$p_s(\mathbf{r}) = i\frac{e^{ikr}}{r}\frac{h_1(kr_0)}{h_1'(ka)}\cos\psi_0,$$

where $\cos\psi_0 = \cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\phi-\phi_0)$.

The total field due to a unit simple source can thus be written, using the asymptotic limit of the Hankel function at small values of the argument

$$p(\mathbf{r}) = \frac{e^{ikR_0}}{R_0}-ik\frac{e^{ikr}}{r}\frac{a^3}{2r_0^2}\cos\psi_0. \tag{5}$$

As the approximations are independent of the source position variable, the dipole field p_d and the longitudinal quadrupole field p_q can be found directly. For radial multipoles at $(r_0, 0, 0)$ the total fields are given by the equations

$$\left.\begin{aligned} p &= \frac{e^{ikr}}{r}-ik\frac{e^{ikr}}{r}\frac{a^3}{2r_0^2}\cos\theta,\\ p_d &= -ik\frac{e^{ikr}}{r}\cos\theta+ik\frac{e^{ikr}}{r}\frac{a^3}{r_0^3}\cos\theta,\\ p_q &= -k^2\frac{e^{ikr}}{r}\cos^2\theta-3ik\frac{e^{ikr}}{r}\frac{a^3}{r_0^4}\cos\theta.\end{aligned}\right\} \tag{6}$$

For compactness, and as these are far-field results, R_0 has been replaced by r.

The corresponding results for a source at $(r_0, \tfrac{1}{2}\pi\ 0)$, when the dipoles and longitudinal quadrupoles have axes parallel to the surface of the sphere, are given by the equations

$$p = \frac{e^{ikr}}{r} - ik\frac{e^{ikr}}{r}\frac{a^3}{2r_0^2}\sin\theta\cos\phi,$$

$$p_d = -ik\frac{e^{ikr}}{r}\cos\theta - ik\frac{e^{ikr}}{r}\frac{a^3}{2r_0^3}\cos\theta,$$

$$p_q = -k^2\frac{e^{ikr}}{r}\cos^2\theta + \tfrac{3}{2}ik\frac{e^{ikr}}{r}\frac{a^3}{r_0^4}\sin\theta\cos\phi.$$

The dipole form of the scattered field is clearly seen. The strength of the scattered field is a factor $O\{(kr_0)^{-n+2}a^3/r_0^3\}$ times the incident field, where n is the order of the source ($n = 1$ corresponds to a simple source). For radially inclined source multipoles the scattering dipole is always inclined in the same direction as the source. However, for sources inclined parallel to the surface of the sphere, the scattering dipole is inclined either in the same direction or along the radius vector to the source as the source number, n, is either even or odd, respectively.

We note that the results demonstrate a particular case of Curle's (1955) analysis of the acoustic radiation from hydrodynamic sources near surfaces. Turbulence can be characterized as a hydrodynamic source by a distribution of quadrupoles. The radiation from these quadrupoles is modified, however, by the presence of surfaces: a dipole component is introduced into the total radiated field. At the low-frequency limit considered here we see that dipole radiation is the dominant feature of the field.

High-frequency limit

At high frequencies ($ka \gg 1$) the surface of the sphere can be divided into a shadow region and an illuminated region. In the shadow region the scattered field almost exactly cancels the incident field. To the degree of approximation considered here the total field in the shadow region is assumed zero: it is, in fact, exponentially small (Morse & Feshbach 1953). The horizons limiting this region are shown in figure 1 when the source is at the point $(r_0, 0, 0)$.

Reflexion at the illuminated surface at this limit is essentially specular reflexion in the tangent plane at the point of reflexion (shown at $\mathbf{r}_1$ in figure 1). The path of the reflected ray R_1 can thus easily be found: we have only to determine the point $\mathbf{r}_1$. For a source at $(r_0, 0, 0)$ the total field is independent of the co-ordinate ϕ. The angle of reflexion is found from the equations

$$\frac{\sin\psi}{r_0} = \frac{\sin(\psi-\theta_1)}{a} = \frac{\sin\theta_1}{R}. \tag{7}$$

where $\psi = \theta - \theta_1$.

We are interested in sources positioned close to the surface. Thus, if $r_0 = a + \delta$, we have $\delta/a \ll 1$. Equation (7) can be solved approximately for small angles θ_1, giving

$$\theta_1 = (\delta/a)\tan\theta.$$

The scattered field is equivalent to a source at the point r_2. Thus the total field is defined by the expression

$$p(\mathbf{r}) = \frac{e^{ikR_0}}{R_0} + \frac{e^{ik(R_1+R)}}{(R_1+R)}$$
$$= \frac{e^{ikR_0}}{R_0}(1+e^{ikR(1+\cos 2\psi)}).$$

A phase shift $kR(1+\cos 2\psi) \approx 2k\delta \sin\theta \tan\theta$ is thus introduced.

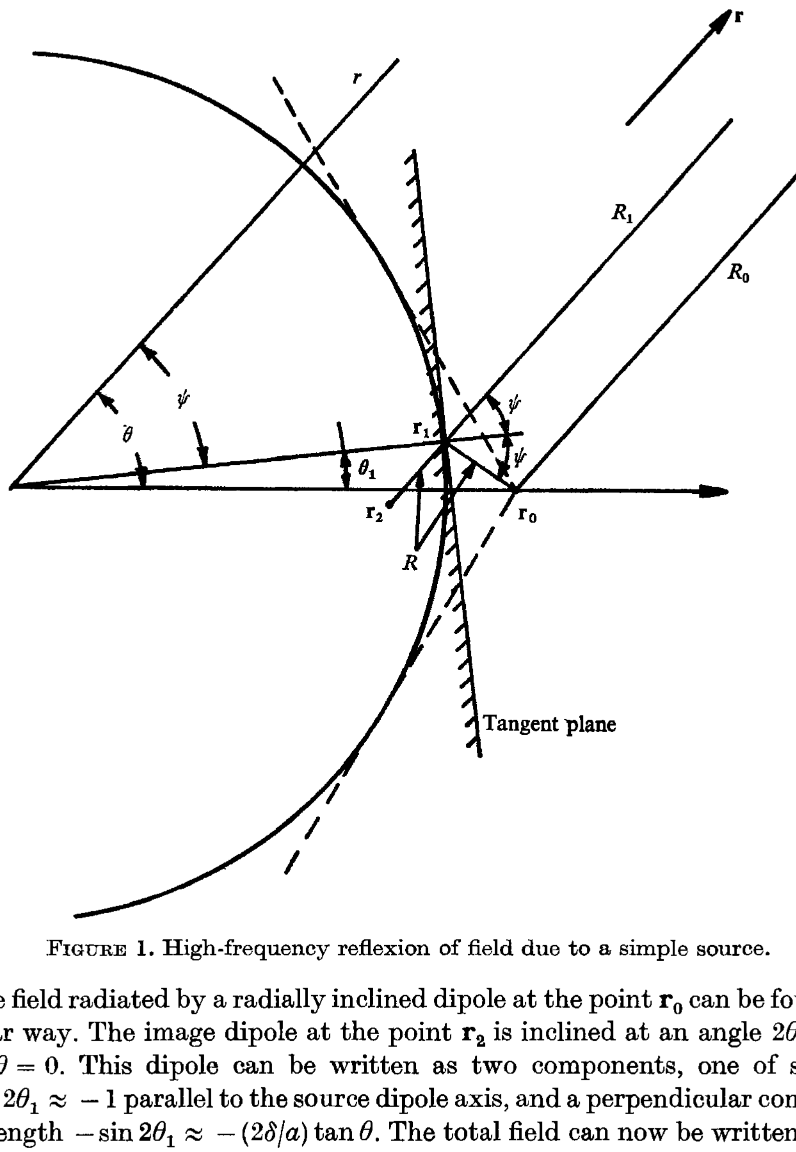

FIGURE 1. High-frequency reflexion of field due to a simple source.

The field radiated by a radially inclined dipole at the point $\mathbf{r}_0$ can be found in a similar way. The image dipole at the point $\mathbf{r}_2$ is inclined at an angle $2\theta_1$ to the axis $\theta = 0$. This dipole can be written as two components, one of strength $-\cos 2\theta_1 \approx -1$ parallel to the source dipole axis, and a perpendicular component of strength $-\sin 2\theta_1 \approx -(2\delta/a)\tan\theta$. The total field can now be written

$$p(\mathbf{r}_1) = ik\cos\theta \frac{e^{ikR_0}}{R_0}(1-e^{2ik\delta \sin\theta\tan\theta}) - 2ik\tan\theta\sin\theta\cos\phi \frac{\delta}{a}\frac{e^{ikR_0}}{R_0}e^{2ik\delta\sin\theta\tan\theta}.$$

The curvature of the scattering body thus introduces a small tangential field component perpendicular to the original dipole axis.

Similarly, a radially inclined longitudinal quadrupole at the point $\mathbf{r}_0$ leads to an image quadrupole at the point $\mathbf{r}_2$ inclined to the $\theta = 0$ axis at an angle $2\theta_1$. Thus, we again obtain a radial quadrupole component of strength 1 and a small tangential quadrupole component of strength $(2\delta/a)\tan\theta$.

The fields of dipoles and longitudinal quadrupoles with axes parallel to the surface of the sphere can be found in the same way. The image multipole is of essentially the same form and strength as the source multipole, but a small radial multipole component of strength $(2\delta/a)\tan\theta$ is now introduced because of the curvature.

These high-frequency results can also be obtained by assuming that on the illuminated part of the sphere the surface pressure field is just twice the incident pressure field. In (3), $p(\mathbf{r}_1)$ can thus be replaced by $2p_i(\mathbf{r}_1)$. The resulting integral can then be evaluated approximately by the method of stationary phase (Davies 1967).

Dipole source near an oscillating sphere

We now apply the low-frequency approximation obtained to the field generated by a dipole source near to an oscillating solid sphere. As the fields generated are linear, the result for an oscillating body can be obtained by adding on the field due to the motion of the body alone. This is true if the oscillations of the body are caused either by the influence of the source alone or by this influence in conjunction with an external force. If the generating source is attached in some way to the sphere, the motion of the source due to this attachment will affect the strength of the source only to a limited extent. This effect is neglected.

Fields due to dipoles at $\mathbf{r}_0$ in the r_0, θ_0, and ϕ_0 directions, can be obtained as above by differentiating (5) by $\partial/\partial r_0$, $(1/r_0)\,\partial/\partial\theta_0$ and $(1/r_0\sin\theta_0)\,\partial/\partial\phi_0$ respectively.

It is convenient to consider the source at the point $(r_0, 0, 0)$. The dipole strengths can be expressed in rectangular axes (x_1, x_2, x_3) chosen to correspond in the usual way to the spherical axes. The dipole strengths A_i corresponding to the r, θ and ϕ directions are thus A_3, A_1 and A_2, respectively. If F_i is the oscillatory force required to generate a dipole of strength A_i, comparing far-field terms shows that $A_i = -F_i/4\pi$. It follows, by differentiating (5), that the total far field due to a dipole near a rigid sphere with components equivalent to forces F_i can be written in the form

$$p(\mathbf{r}) = \frac{-ik}{4\pi}\frac{e^{ikr}}{r}\left\{\begin{aligned} &F_1\left(1+\frac{a^3}{2r_0^3}\right)\sin\theta\cos\phi \\ &\qquad +F_2\left(1+\frac{a^3}{2r_0^3}\right)\sin\theta\sin\phi \\ &\qquad +F_3\left(1-\frac{a^3}{r_0^3}\right)\cos\theta. \end{aligned}\right\}. \tag{8}$$

If the radial velocity of the sphere, assumed small and denoted by $U(\theta_0, \phi_0)$, is known, the low-frequency far-field pressure generated by the motion can be found using an analysis similar to the above. As the sphere is rigid, the motion can be described by specifying three constant oscillations in three mutually perpendicular directions. Choosing these directions as the Cartesian co-ordinates already defined and denoting the velocity in the i direction by U_i, the radial velocity can be written

$$U(\theta_0, \phi_0) = U_1 \sin\theta_0 \cos\phi_0 + U_2 \sin\theta_0 \sin\phi_0 + U_3 \cos\theta_0.$$

The low-frequency pressure field is easily found to be

$$p(\mathbf{r}) = -\frac{\omega k}{4\pi}\frac{M_a}{2}\frac{e^{ikr}}{r}(U_1 \sin\theta\cos\phi + U_2 \sin\theta\sin\phi + U_3\cos\theta), \tag{9}$$

where M_a is written for the virtual mass $\frac{4}{3}\pi a^3\rho$. The total pressure at large distances is the sum of the pressures defined by (8) and (9).

The magnitudes of the velocities U_i can be found from the equation of motion of the sphere. If G_i is the external applied force in the i direction and M is the mass of the sphere, the equation of motion is

$$G_i = \int_s p(\mathbf{r}_1)\, l_i \, d\mathbf{S}(\mathbf{r}_1) + M\,\partial U_i/\partial t.$$

l_i is the direction cosine of the radius vector to $\mathbf{r}_1$ to the i rectangular axis. The pressure under the integral is the total surface pressure due to the source field and the motion of the sphere. The integration is similar to that already performed. Thus, for example, the oscillating velocity in the 3 direction is given by the equation

$$U_3 = \frac{i\{G_3 + F_3(a^3/r_0^3)\}}{\omega(M+\frac{1}{2}M_a)}.$$

We note that the effective mass term $(M+\frac{1}{2}M_a)$ includes the effect of the virtual inertia of the sphere. Similar results hold for U_1 and U_2. Using these values of the velocity and equations (8) and (9) the total field generated by the system being considered is

$$p(\mathbf{r}) = \frac{ik}{4\pi}\frac{e^{ikr}}{r}\left\{\begin{aligned} &\left[F_1\left(1+\frac{M}{M+\frac{1}{2}M_a}\frac{a^3}{2r_0^3}\right)+\frac{\frac{1}{2}M_a}{M+\frac{1}{2}M_a}G_1\right]\sin\theta\cos\phi \\ &+\left[F_2\left(1+\frac{M}{M+\frac{1}{2}M_a}\frac{a^3}{2r_0^3}\right)+\frac{\frac{1}{2}M_a}{M+\frac{1}{2}M_a}G_2\right]\sin\theta\sin\phi \\ &+\left[F_3\left(1-\frac{M}{M+\frac{1}{2}M_a}\frac{a^3}{r_0^3}\right)+\frac{\frac{1}{2}M_a}{M+\frac{1}{2}M_a}G_3\right]\cos\theta. \end{aligned}\right\}. \tag{10}$$

Two limiting cases can be noted. If the sphere has infinite mass the solution reduces to the fixed sphere result. If the mass of the sphere is zero (e.g. an air bubble in water) the sound generated is equivalent to dipoles of strength $F_i + G_i$. It follows immediately from equation (10) that the sound generated by the system can be reduced by choosing suitable values of G_i. If the dipole generating mechanism is attached to the body the least value of the radiated sound field is obtained when the dipole is rigidly attached, i.e. when $G_1 = G_2 = 0$ and $G_3 = -F_3$. It can also be seen that the sound pressure radiated in this case is less than that radiated when the sphere is fixed.

Summary and conclusions

High- and low-frequency approximations have been obtained for the acoustic fields radiated by simple point sources and point multipoles near the surface of a rigid sphere. It is convenient to note here the conditions under which the high-frequency results are valid. It is necessary that $kr \gg ka \gg 1$. Also, the approximation breaks down if the point of reflexion $\mathbf{r}_1$ is at the horizon. Thus the result gives the leading term for points of reflection well within the illuminated region. At high frequencies this region contains the major part of the radiation: the acoustic field in the shadow region is exponentially small.

The results can be used to estimate the noise generated by a turbulent boundary layer on a rigid sphere. It is of interest to compare features of the sound field radiated at each limit. The turbulence is basically a quadrupole source. Meecham (1965) has suggested that the scattered field in this case is equivalent to an image source distribution differing from the actual source distribution by the factor δ/a, where δ is the boundary-layer thickness. However, the results obtained above demonstrate that an important parameter in the problem is ka. Meecham's use of the result is only justified at high values of ka. At low frequencies quadrupole radiation is enhanced by a scattered dipole field. The radiation efficiency of the turbulence is thus considerably increased. At high frequencies the total radiated field is of quadrupole type and the increase in radiation efficiency noted at low frequencies does not exist.

As previously mentioned, the results demonstrate a particular case of Curle's (1955) analysis. The total radiated field consists of direct radiation from the volume of quadrupoles characterizing the turbulence, together with the so-called surface sound of essentially dipole form. The relative magnitudes of the two forms of radiation depend, in the case considered here, on the parameter ka. In a turbulent boundary layer, typical values of the frequency can be related to a Strouhal number S based on a mean flow velocity. Incorporating S into the ratio ka shows that the relevant limit is determined by the value of the parameter MS, where M is the Mach number of the mean flow. For large values of MS the dominant radiation is quadrupole. For small values of MS the surface sound radiated pressure is a factor

$$\left\{(kr_0)^{-1}\frac{a^3}{r_0^3}\right\} \propto (MS)^{-1}$$

times the quadrupole radiation. It follows that at low Mach numbers, for example, in water, where typical Mach numbers are very low indeed, the radiated sound is characteristically dipole.

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